Playing dominos in different domains

Marianna Russkikh
A dimer covering or a perfect matching of a graph is a subset of edges that covers every vertex exactly once. On a square lattice it can be viewed as a random tiling of a domain $\Omega$ on the dual lattice by dominos $2 \times 1$.

How does a generic one look? (e.g. correlations of dominos)
Domino tilings

**NON LIQUID** case → must connect with other methods

**LIQUID** case
Thurston introduced the height function of a tiling which uniquely assigns integer values to all vertices. Vice versa, a domino tiling can be reconstructed from the values of the height function. Thus one can think of a random domino tiling as a random height function on the vertex set of the domain.

**Dimer height function** on vertices: along each edge not covered by a domino the height changes by ±1, increases by 1 if this edge has a black square on its left, and decreases by 1 otherwise.

The key questions: the large-scale behavior of

(a) the expectation of the height function,

(b) fluctuations of the height function.
Lozenge tilings can in turn be viewed as orthogonal projections onto the plane $P_{111} = \{x + y + z = 0\}$ of stepped surfaces which are polygonal surfaces in $\mathbb{R}^3$ whose faces are squares in the 2-skeleton of $\mathbb{Z}^3$.

Height function is equal to $\sqrt{3}$ times the distance from the surface to the plane $P_{111}$. 

$h(z_0) = 0$ 

$1 = h(z)$
Lozenge tiling
Kasteleyn matrix

\[
K_\Omega = \begin{pmatrix}
  w_1 & w_2 & w_3 \\
  b_1 & 1 & 1 & 0 \\
  b_2 & -1 & 1 & -1 \\
  b_3 & 0 & 1 & 1 
\end{pmatrix}
\]

A Kasteleyn matrix $K_\Omega$ is a weighted adjacency matrix whose rows index the black vertices and columns index the white vertices.

**Theorem (Percus'69, Kasteleyn'61)**

\[
\# \text{ domino tilings} = |\det K_\Omega|
\]

**Corollary**

\[
\left| \frac{\det(K_\Omega \setminus \{u,v\})}{\det(K_\Omega)} \right| = |K_\Omega^{-1}(u, v)|
\]
Kasteleyn matrix

More symmetric $\mathbb{C}$ weights proposed by Kenyon:

Relation for 4 values of $K^{-1}_\Omega$:

$$1 \cdot K^{-1}_\Omega(v + 1, v') - 1 \cdot K^{-1}_\Omega(v - 1, v') +$$

$$i \cdot K^{-1}_\Omega(v + i, v') - i \cdot K^{-1}_\Omega(v - i, v') = \delta_{\{v=v'\}}$$
Kasteleyn matrix

More symmetric \( \mathbb{C} \) weights proposed by Kenyon:

\[
K_{\Omega}^{-1} \times K_{\Omega} = \text{Id}
\]

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\]

Discrete Cauchy-Riemann:

\[
F(c) - F(a) = -i \cdot (F(d) - F(b))
\]
Inverse Kasteleyn matrix

Let us denote by $C_\Omega(u, \nu)$ the elements of the inverse matrix $K_\Omega^{-1}$, where $u$ and $\nu$ are black and white squares of $\Omega$.

For any fixed white square $\nu_0 \in \diamondsuit_0$

\[
\begin{align*}
C_\Omega(u, \nu_0) & \in \mathbb{R}, & u & \in \diamondsuit_0, \\
C_\Omega(u, \nu_0) & \in i\mathbb{R}, & u & \in \diamondsuit_1.
\end{align*}
\]

The function $C_\Omega(u, \nu)$ is called the coupling function.
Inverse Kasteleyn matrix

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For any fixed white square $\nu_0 \in \blacklozenge_0$

\[
\begin{cases}
C_{\Omega}(u, \nu_0) \in \mathbb{R}, & u \in \blacklozenge_0, \\
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\end{cases}
\]

The function $C_{\Omega}(u, \nu)$ is called the **coupling function**.
Coupling function

Two main properties of the coupling function $C_\Omega: \Diamond \times \Diamond \rightarrow \mathbb{C}$ are the following:

- if $\nu \in \Diamond$, then $C_\Omega(u, \nu)$ is a discrete holomorphic function of $u$ with a simple pole at $\nu$;
- if $u$ and $\nu$ are adjacent squares, then $|C_\Omega(u, \nu)|$ is equal to the probability that the domino $[uv]$ is contained in a random domino tiling of $\Omega$.

Kenyon used this approach to prove the conformal invariance of the limiting distribution of the height function in the case of Temperley discretisations.
Coupling function

Two main properties of the coupling function $C_\Omega: \bar{\diamond} \times \bar{\diamond} \rightarrow \mathbb{C}$ are the following:

- if $v \in \diamond$, then $C_\Omega(u, v)$ is a discrete holomorphic function of $u$ with a simple pole at $v$;
- if $u$ and $v$ are adjacent squares, then $|C_\Omega(u, v)|$ is equal to the probability that the domino $[uv]$ is contained in a random domino tiling of $\Omega$.

Kenyon used this approach to prove the conformal invariance of the limiting distribution of the height function in the case of Temperley discretisations.

$$
\mathbb{E}(h(z_1) - h(z_2)) = 3 \cdot P[uv] - 1 \cdot (1 - P[uv]) = 4 \cdot |C_\Omega(u, v)| - 1
$$
Discrete holomorphicity

\[ F(c) - F(a) = -i \cdot (F(d) - F(b)) \]

\[ 4[\Delta F](u) = \sum_{s=1}^{4} (F(u_s) - F(u)) \]

Real and imaginary parts of a holomorphic function are harmonic functions. It is also true on the discrete level. So, real and imaginary parts of the coupling function are discrete harmonic functions.
Discrete holomorphicity

**Discrete Cauchy-Riemann:**

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A domain in which all corner squares are dark grey is called an *odd Temperley domain*. To obtain a *Temperley domain* one removes a dark grey square adjacent to the boundary from an odd Temperley domain.
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Known results

**Theorem (Kenyon, 1999)**

Let $\Omega$ be a bounded, simply connected domain in $\mathbb{C}$. Assume that a sequence of discrete Temperley domains $\Omega^\delta$ on a grid with mesh size $\delta$ approximates the domain $\Omega$. Then coupling functions $\frac{1}{\delta} C_{\Omega^\delta}(u, v)$ converge to the derivative of Green’s function.

**Theorem (Kenyon, 2000)**

If one considers Temperley discretisations $\Omega^\delta$ of a given domain $\Omega$, then the fluctuations of the height function converge (as the mesh size $\delta$ tends to zero) to the Gaussian Free Field on $\Omega$ with Dirichlet boundary conditions.
Boundary conditions

The coupling function $C_{\Omega^\delta}(u, v)$ restricted to the light grey squares satisfies the Dirichlet boundary conditions, and $C_{\Omega^\delta}(u, v)$ restricted to the dark grey squares obeys Neumann boundary conditions.
We call a discrete domain a 2n-black-piecewise Temperley domain if it has $n + 1$ convex white corners and $n - 1$ concave white corners.
Boundary conditions

Mixed Dirichlet and Neumann boundary conditions for the coupling function
Hedgehog domains

This picture corresponds to more involved boundary conditions than mixed Dirichlet and Neumann ones.
Hedgehog domains
Hedgehog domains
[Kenyon, 99]

**Open Question**

$$F(z) \| \frac{1}{\sqrt{n(z)}}$$
Boundary height
**Results**

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A double-dimer configuration is a union of two dimer coverings or equivalently a set of even-length simple loops and double edges with the property that every vertex is the endpoint of exactly two edges. By orienting the edges of the first covering from white to black and the edges of the second one from black to white, one gets an orientation of resulting loops.
Double dimers

One can consider coverings of a pair of domains that differ on two squares. In this case, in addition to a collection of loops and double edges, the resulting superposition of these coverings contains an interface (a simple path between these two squares).
Double-dimer height function

The double-dimer height function on $\Omega = \Omega_1 \cup \Omega_2$:

$$H_{d-d,\Omega}(z) := H_{\Omega_1}(z) - H_{\Omega_2}(z).$$

$H_{d-d,\Omega}$ has a simple geometric representation: a cross a path changes the height function by $+4$ or $-4$, depending on the orientation of the path.
Double dimers

The double-dimer coupling function on $\Omega = \Omega_1 \cup \Omega_2$:

$$C_{d-d,\Omega}(u, v) := C_{\Omega_1}(u, v) - C_{\Omega_2}(u, v).$$
Results

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Factorization

\[
\begin{align*}
Z & = Z \cdot K^{-1}(u_0, v_0) \\
Z & = Z \cdot K^{-1}(u, v) \\
Z & = Z \cdot \begin{vmatrix} K^{-1}(u_0, v_0) & K^{-1}(u_0, v) \\ K^{-1}(u, v_0) & K^{-1}(u, v) \end{vmatrix}
\end{align*}
\]

\[
\begin{bmatrix} P & - P \end{bmatrix} \begin{bmatrix} - \\ - \end{bmatrix} = \frac{Z}{Z} - \frac{Z}{Z} = \frac{K^{-1}(u, v_0) \cdot K^{-1}(u_0, v)}{K^{-1}(u_0, v_0)}
\]
The primitive of the product of two discrete holomorphic functions

Let $F : \diamondsuit \rightarrow \mathbb{C}$ and $G : \heartsuit \rightarrow \mathbb{C}$ be discrete holomorphic functions. Let $H : \mathcal{V} \rightarrow \mathbb{R}$ be defined by

$$H(z_2) - H(z_1) = (z_2 - z_1)F(u)G(v),$$

where $z_1, z_2$ are two common vertices of a pair of adjacent black and white squares $u, v$.

1. The function $H(\cdot)$ is a well-defined real-valued function which does not have local extrema.

2. The coupling function in the double-dimer model is a product of two discrete holomorphic functions:

$$C_{d-d}(b^\bullet, w^\circ) = \text{const} \cdot F(b^\bullet) \cdot G(w^\circ).$$

3. The expectation of the height function coincides with the function $H$:

$$\mathbb{E}[h_{d-d}(z)] = \text{const} \cdot H(z).$$
Fix two black squares $u_1, u_2 \in \partial_{\text{int}}\blacklozenge$ in such a way, that after removing one of them the resulting domain allows a domino tiling. Then the following holds:

1. There exists a unique function $F : \blacklozenge \to \mathbb{C}$ such that $F|_{\partial\blacklozenge} = 0$, $F(u_1) = 1$ and $F$ is discrete holomorphic everywhere in $\blacklozenge$.

2. There exists a unique function $G : \blacklozenge \to \mathbb{C}$ such that $G|_{\partial\blacklozenge} = 0$ and $G$ is discrete holomorphic everywhere in $\blacklozenge$ except at faces $u_1, u_2$ and one has $[\partial G](u_1) = i$. 
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Leap-frog harmonicity

Kenyon proved that for the single-dimer model, the height function expectation is harmonic in the limit for approximations by Temperley domains.

\[ \text{Leap-frog Laplacian} \]

\[
[L_f H](z) = \sum_{z_s \sim z} \frac{H(z_s) - H(z)}{4}
\]

Using the above factorization of the double-dimer coupling function we show that the difference of the expectations of two height functions is harmonic already on a discrete level.

**Theorem (R., 2016)**

The expectation of the double-dimer height function on a Temperley domain is exactly discrete leap-frog harmonic.
Leap-frog harmonicity

Computation:

\[
\delta^{-1} \cdot [\Delta_{L_f} H](z) = \\
\lambda [\partial F](d)[\partial G](a) - \bar{\lambda}[\partial F](d)[\partial G](c) \\
+ \bar{\lambda}[\partial F](b)[\partial G](a) - \lambda[\partial F](b)[\partial G](c)
\]
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\[ + \bar{\lambda} [\partial F](b)[\partial G](a) - \lambda [\partial F](b)[\partial G](c) \]

Let \( \Omega^\delta \) be Temperley domain:

1. \( F \) is a discrete holomorphic function at all white squares.
2. So, its imaginary part is a discrete harmonic function with zero boundary conditions.
3. Therefore \( \text{Im} F \) is identically zero, and thus \( \text{Re} F \) is a constant. Hence, \( \partial F \) is identically zero.
# Results

## Dimer model

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## Double-dimer model

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| $\mathbb{E}H^\delta_{d-d}$ | $\Delta^\delta[\mathbb{E}H^\delta_{d-d}] = 0$ | $\mathbb{E}H^\delta_{d-d} \to h_{\Omega}^{(ab)}(\cdot)$ | $F^\delta \to f_{\Omega}$ | $\mathbb{E}H^\delta_{d-d}(\cdot) \to h_{\Omega}^{(ab)}(\cdot)$ |
Let a domain $\Omega$ contain the same number of black and white squares, and let $\Omega$ allow a domino tiling. Fix a black square $u_0 \in \partial_{\text{int}} \diamond$ and a white square $v_0 \in \partial_{\text{int}} \lozenge$ such that the domain $\Omega \setminus \{u_0, v_0\}$ allows a domino tiling. Then the following holds:

1. There exists a unique function $F : \diamond \to \mathbb{C}$ such that $F|_{\partial \diamond} = 0$ and $F$ is discrete holomorphic everywhere in $\lozenge$ except at the face $v_0$ where one has $[\bar{\partial} F](v_0) = \lambda$. Moreover, $F(u_0) \neq 0$.

2. There exists a unique function $G : \lozenge \to \mathbb{C}$ such that $G|_{\partial \lozenge} = 0$ and $G$ is discrete holomorphic everywhere in $\diamond$ except at $u_0$ where one has $[\bar{\partial} G](u_0) = i$. Moreover, $G(v_0) \neq 0$. 
Continuous analogues of the functions $F^\delta$ and $G^\delta$

- $f_\Omega(z) = \frac{\lambda}{z-v_0} + O(1)$ in a vicinity of the point $v_0$;
- $f_\Omega$ is bounded in vicinities of the points $v_k^*$;
- $f_\Omega$ is semi-bounded in vicinities of the points $\tilde{v}_k$;
- along each boundary arc between marked points $\{v_k^*\}_{k=1}^{n+1} \cup \{\tilde{v}_k\}_{k=1}^{n-1}$, one has either $\text{Re}[f_\Omega] = 0$ or $\text{Im}[f_\Omega] = 0$;
- boundary conditions change at all marked points $\tilde{v}_k$ and $v_s^*$. 
Continuous analogues of the functions $F^\delta$ and $G^\delta$

Let $\phi$ be a conformal mapping of the domain $\Omega$ onto the upper half plane $\mathbb{H}$ such that none of the marked points are mapped onto infinity, then

$$f_{\Omega}(z) = \frac{c_\phi}{(\phi(z) - \phi(v_0))} \cdot \prod_{k=1}^{n+1} (\phi(z) - \phi(v_k^*))^{\frac{1}{2}} \cdot \prod_{k=1}^{n-1} (\phi(z) - \phi(\tilde{v}_k))^{-\frac{1}{2}},$$

where $c_\phi$ is a real constant depending on $\phi$:

$$c_\phi = \lambda \cdot \phi'(v_0) \cdot \prod_{k=1}^{n+1} (\phi(v_0) - \phi(v_k^*))^{-\frac{1}{2}} \cdot \prod_{k=1}^{n-1} (\phi(v_0) - \phi(\tilde{v}_k))^{\frac{1}{2}}.$$
Piecewise Temperley domains

Theorem (R., 2016)

Let $\Omega^\delta$ be a sequence of discrete black-piecewise Temperley
domains of mesh size $\delta$ approximating a continuous domain $\Omega$.
Suppose that each $\Omega^\delta$ allows a domino tiling. Then $F^\delta$ converge
to $f_\Omega$ uniformly on compact subsets of $\Omega$.

Theorem (R., 2016)

Let $\Omega^\delta$ be a piecewise Temperley domain approximating $\Omega$. Let $h^\delta$
be the height function of $\Omega^\delta$ and $\overline{h^\delta}$ be its mean value. Then, as $\delta$
tends to 0, $h^\delta - \overline{h^\delta}$ converges weakly in distribution to the
Gaussian Free Field on $\Omega$ with Dirichlet boundary conditions.
Piecewise Temperley domains

Theorem (R., 2016)

Let $\Omega^\delta$ be a sequence of discrete **black-piecewise Temperley domains** of mesh size $\delta$ approximating a continuous domain $\Omega$. Suppose that each $\Omega^\delta$ allows a domino tiling. Then $F^\delta$ converge to $f_\Omega$ uniformly on compact subsets of $\Omega$.

1. Note that the coupling function $C^\delta(u, v_0)$ with fixed $v_0$ coincides with a discrete holomorphic function $F^\delta(u)$.
2. Similarly, one can show the convergence of $G^\delta$ for approximations by **white-piecewise Temperley** domains:

$$g_\Omega(z) = \frac{\lambda c_\phi}{(\phi(z) - \phi(u_0))} \cdot \prod_{k=1}^{m+1} (\phi(z) - \phi(u_k^*))^{\frac{1}{2}} \cdot \prod_{k=1}^{m-1} (\phi(z) - \phi(\tilde{u}_k))^{-\frac{1}{2}}.$$


A polygonal domain $\Omega^\delta$ is black-piecewise Temperley and also white-piecewise Temperley. So, we obtain the convergence of the double-dimer coupling function for polygonal domain.

The product of the functions $f_\Omega(z)$ and $g_\Omega(z)$ does not depend on the colours of corners of $\Omega$, while each of $f_\Omega(z)$, $g_\Omega(z)$ does depend on these colours.
Theorem (R., 2016)

Let $\Omega$ be a polygon with $n$ sides parallel to the axes. Let $u_0$ and $v_0$ be points on straight parts of the boundary of domain $\Omega$. Suppose that a sequence of discrete $n$-gons $\Omega^\delta$ on a grid with mesh size $\delta$ approximates the polygon $\Omega$ in a proper way. Let black and white squares $u_0^\delta$ and $v_0^\delta$ of domain $\Omega^\delta$ tend to boundary points $u_0$ and $v_0$ of the domain $\Omega$. Let $h^\delta$ be a double-dimer height function on $\Omega$. Then $Eh^\delta$ converges to the harmonic measure $hm_{\Omega}(\cdot,(u_0v_0))$ of the boundary arc $(u_0v_0)$ on the domain $\Omega$. 
# Results

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<td>$H^\delta - \mathbb{E}H^\delta$ converges to GFF with Dirichlet boundary conditions</td>
<td></td>
</tr>
</tbody>
</table>

### Double-dimer model

<table>
<thead>
<tr>
<th>$C_{\Omega}^{d-d}(u, v)$</th>
<th>$C_{\Omega}^{d-d}(u^<em>, v^\circ) = \text{const} \cdot F^\delta(u^</em>) \cdot G^\delta(v^\circ)$, where $F^\delta$ and $G^\delta$ are discrete holomorphic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E}H_{d-d}^\delta$</td>
<td>$\Delta^\delta[\mathbb{E}H_{d-d}^\delta] = 0$</td>
</tr>
</tbody>
</table>
Hedgehog domains

A discrete simply connected domain $\Omega$ is called an *Hedgehog* domain if it composed of a finite number of squares $2 \times 2$ and for each such a square touching the boundary exactly two of its adjacent sides belong to the boundary of $\Omega$. 
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Hedgehog domains

\[ F_{s-hol}(u) := \frac{1}{\delta} C_{\Omega^\delta}(u, v_0) \text{ for any } u \in \Omega \]

Theorem (R., 2017)

Let \( \Omega^\delta \) be a sequence of discrete hedgehog domains of mesh size \( \delta \) approximating a continuous domain \( \Omega \). Let \( v_0^\delta \) approximate a point \( v \in \Omega \). Then \( F_{s-hol} \) converges uniformly on compact subsets of \( \Omega \setminus \{v\} \) to a holomorphic function \( f^v_\Omega \), where \( f^v_\Omega \) satisfies (f1)–(f2).

Theorem (R., 2017)

Let \( \Omega \) be a Jordan domain with smooth boundary in \( \mathbb{R}^2 \). Let \( \Omega^\delta \) be an hedgehog domain approximating \( \Omega \). Let \( h^\delta \) be the height function of \( \Omega^\delta \) and \( \overline{h^\delta} \) be its mean value. Then \( h^\delta - \overline{h^\delta} \) converges weakly in distribution to the Gaussian Free Field on \( \Omega \) with Dirichlet boundary conditions, as \( \delta \) tends to 0.
Continuous analogue of functions $F_{s-hol}^\delta$

Let $\Omega$ be a bounded (simply connected) domain with smooth boundary, and $\nu \in \Omega$. Then there exists a unique holomorphic function $f^\nu_\Omega$ such that:

(f1) $f^\nu_\Omega(z) = \frac{1}{2\pi} \cdot \frac{\lambda}{z - \nu} + O(1)$ in a vicinity of the point $\nu$;

(f2) $\left| f^\nu_\Omega(z) \right| \left| \frac{1}{\sqrt{n(z)}} \right|$, $z \in \partial \Omega$. 

Let $\psi$ be a conformal mapping of the domain $\Omega$ onto the unit disk $D$ such that $\nu$ mapped onto 0 and $\psi'(0) > 0$. Then

$f^\nu_\Omega(z) = f^0_\Omega(D(\psi(z))) \cdot (\psi'(0))^\frac{1}{2} \cdot (\psi'(\nu))^\frac{1}{2}$,

where $f^0_\Omega = \frac{1}{2\pi} \cdot \frac{\lambda}{z + \bar{\nu}}$ is a solution in the unit disk with singularity at zero.
Let $\Omega$ be a bounded (simply connected) domain with smooth boundary, and $\nu \in \Omega$. Then there exists a unique \textit{holomorphic} function $f_{\Omega}^\nu$ such that:

(f1) $f_{\Omega}^\nu(z) = \frac{1}{2\pi} \cdot \frac{\lambda}{z-\nu} + O(1)$ in a vicinity of the point $\nu$;

(f2) $f_{\Omega}^\nu(z)||\frac{1}{\sqrt{n(z))}}, \quad z \in \partial \Omega$.

Let $\phi$ be a conformal mapping of the domain $\Omega$ onto the unit disk $\mathbb{D}$ such that $\nu$ mapped onto 0 and $\phi'(\nu) > 0$. Then

$$f_{\Omega}^\nu(z) := f_{\mathbb{D}}^0(\phi(z)) \cdot (\phi'(z))^{\frac{1}{2}} \cdot (\phi'(\nu))^{\frac{1}{2}},$$

where $f_{\mathbb{D}}^0 = \frac{1}{2\pi} (\frac{\lambda}{z} + \bar{\lambda})$ is a solution in the unit disk with singularity at zero.
Critical Ising model on \( \bullet \) (black vertices)

[Smirnov, Chelkak-Smirnov, Chelkak-Hongler-Izyurov...]

* s-holomorphic observables

A function \( F_{\text{hol}} \mid V \) is s-holomorphic if for each pair of vertices \( z_1, z_2 \in V \) of the same square \( p \)

\[
\text{Proj}_{\tau(p)}[F_{\text{hol}}(z_1)] = \text{Proj}_{\tau(p)}[F_{\text{hol}}(z_2)],
\]

where \( \text{Proj}_{\tau(p)}[z] = \tau(p) \cdot \text{Re} \left[ z \cdot \overline{\tau(p)} \right] \)

and \( \tau(p) \) is 1, \( i \), \( \lambda \) or \( \bar{\lambda} \).

**Riemann-type** boundary conditions: \( F_{\text{hol}}(z) \mid |\frac{1}{\sqrt(n(z))}| \).
Let $F : \bar{\diamond} \to \mathbb{C}$ be discrete holomorphic. Define $F_{s-hol}$ as follows:

\[
\begin{align*}
F_{s-hol}(u) &= F(u) & \text{if } u \in \diamond ; \\
F_{s-hol}(z) &= F(u_R) + F(u_I) & \text{if } z \in \mathcal{V}_\diamond ; \\
F_{s-hol}(v_\lambda) &= \text{Proj}_\lambda[F_{s-hol}(z)] & \text{if } v_\lambda \in \diamond_0 ; \\
F_{s-hol}(v_\bar{\lambda}) &= \text{Proj}_{\bar{\lambda}}[F_{s-hol}(z)] & \text{if } v_\bar{\lambda} \in \diamond_1 ,
\end{align*}
\]
The primitive of the square of s-holomorphic function

Let $H_{s\text{-hol}} : \mathcal{V}_\bullet \square \mathcal{V}_\circ \to \mathbb{R}$ be a function defined by the equality

$$H_{s\text{-hol}}^\bullet(z_2) - H_{s\text{-hol}}^\circ(z_1) = F_{s\text{-hol}}^2(a) \cdot (z_2 - z_1),$$

where $z_1 \in \mathcal{V}_\circ$, $z_2 \in \mathcal{V}_\bullet$ are two vertices of the same square $a$.

1. $H_{s\text{-hol}}(\cdot)$ is a well-defined real-valued function.
2. $H_{s\text{-hol}}^\bullet(z) \leq H_{s\text{-hol}}^\circ(z')$ for adjacent $z$ and $z'$.
3. $H_{s\text{-hol}}^\bullet$ is leap-frog superharmonic on $\mathcal{V}_\bullet$.
4. $H_{s\text{-hol}}^\circ$ is leap-frog subharmonic on $\mathcal{V}_\circ$. 
Let $H_{s-hol}: \mathcal{V}_\bullet \sqcup \mathcal{V}_\circ \to \mathbb{R}$ be a function defined by the equality

$$H^\bullet_{s-hol}(z_2) - H^\circ_{s-hol}(z_1) = F^2_{s-hol}(a) \cdot (z_2 - z_1),$$

where $z_1 \in \mathcal{V}_\circ$, $z_2 \in \mathcal{V}_\bullet$ are two vertices of the same square $a$.

1. $H_{s-hol}(\cdot)$ is a well-defined real-valued function.
2. $H^\bullet_{s-hol}(z) \leq H^\circ_{s-hol}(z')$ for adjacent $z$ and $z'$.
3. $H^\bullet_{s-hol}$ is leap-frog superharmonic on $\mathcal{V}_\bullet$.
4. $H^\circ_{s-hol}$ is leap-frog subharmonic on $\mathcal{V}_\circ$. 
Boundary conditions of the coupling function in hedgehog domains

Fix a white square \( v_0 \in \text{Int} \diamond_0^\delta \). Let \( F(u) := \frac{1}{\delta} C_{\Omega^\delta}(u, v_0) \). Note that \( F \) is a discrete holomorphic function, with a simple pole at \( v_0 \), therefore one can define \( F_{\text{s-hol}} \) on the set \( (\overline{\Omega}^\delta \setminus \{v_0\}) \cup \mathcal{V}_\diamond^\delta \).

\[
-ic_{\Omega^\delta}(u_l, v_0) + c_{\Omega^\delta}(u_R, v_0) = 0
\]

\[
\Downarrow
\]

\[
F_{\text{s-hol}}^2(u_l) = -F_{\text{s-hol}}^2(u_R)
\]

\[
\Downarrow
\]

\[
H_{\text{s-hol}}^\circ(z_1) = H_{\text{s-hol}}^\circ(z_2)
\]

\( H_{\text{s-hol}}^\circ \) satisfies Dirichlet b.c.: \( H_{\text{s-hol}}^\circ(z) = 0 \) for any \( z \in \partial \mathcal{V}_\diamond^\delta \).

For any \( z \in \partial \mathcal{V}_\diamond^\delta \) one has \( F_{\text{s-hol}}(z)\|\frac{1}{\sqrt{n(z)}} \).
Thank you!

<table>
<thead>
<tr>
<th>Temperley</th>
<th>Piecewise Temperley</th>
<th>Hedgehog</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Re} F = 0$</td>
<td>$\text{Re} F = 0$, $\text{Im} F = 0$</td>
<td>$F(z) \parallel \frac{1}{\sqrt{n(z)}}$</td>
</tr>
<tr>
<td>$C_{\Omega}^d(\cdot, \cdot) \to f_{\text{Temp}}(\cdot, \cdot)$</td>
<td>$C_{\Omega}^d(\cdot, \cdot) \to f_{\text{P-Temp}}(\cdot, \cdot)$</td>
<td>$C_{\Omega}^d(\cdot, \cdot) \to f_{\text{Hedgehog}}(\cdot, \cdot)$</td>
</tr>
</tbody>
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$H^d - \mathbb{E}H^d$ converges to GFF with Dirichlet boundary conditions

\[
C_{\Omega}^{d-d}(u^\bullet, v^\circ) = \text{const} \cdot F^d(u^\bullet) \cdot G^d(v^\circ) \quad \Rightarrow \quad \Delta^d[\mathbb{E}H^{d-d}] \equiv 0
\]