Riemann mapping theorem and Green function

**Riemann mapping theorem.** Suppose $\Omega$ is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $F: \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.

Assume that the boundary of domain $\Omega$ is a Jordan curve. Then any conformal uniformization $F$ is continuous up to the boundary.

**Green function.** Let $\Omega \subset \mathbb{C}$ be a bounded domain and $z_0 \in \Omega$. A function $G(z, z_0)$ defined for all $z \in \mathbb{C} \setminus \{z_0\}$ is called the Green function if the following holds:

- $G(\cdot, z_0)$ is harmonic on $\Omega \setminus \{z_0\}$ and continuous up to the boundary;
- $G(z, z_0) = 0$ if $z \in \partial \Omega$;
- $G(z, z_0) = -\log|z - z_0| + O(1)$ if $z \to z_0$.

The main message: assume that $\Omega$ is a Jordan domain and $z_0 \in \Omega$. Then the existence of the uniformization map $F$ is equivalent to the existence of the Green function $G(\cdot, z_0)$.

$Riemann\ mapping\ theorem \implies \ existence\ of\ the\ Green\ function$

1. Check that $G(x, 0) := \log|x|$ is the Green function on the unit disc. For any $a \in \mathbb{D}$ find $G(x, a)$ using automorphisms of $\mathbb{D}$.

2. Using Riemann mapping theorem show that a Green function exists for any domain $\Omega$.

$Existence\ of\ the\ Green\ function \implies \ Riemann\ mapping\ theorem$

3. Let $\Omega \subset \mathbb{C}$ be a Jordan domain, $z_0 \in \Omega$ and $G(z) = G(z, z_0)$ be the Green function. Define $u(z) = G(z, z_0) + \log|z - z_0|$.

   a) Let $\tilde{G}$ be a harmonic conjugate to $G$ (defined on $\Omega \setminus \{z_0\}$). Note that $\tilde{G}$ is well-defined only locally, it has a branching point $z_0$. Show that

   $$\tilde{G}(z) = -\arg(z - z_0) + \tilde{u}(z),$$

   where $\tilde{u}$ is a harmonic conjugate of $u$.

   b) Set $F(z) = \exp(-G(z) - i\tilde{G}(z))$. Check that $F(z)$ is a well-defined holomorphic function on $\Omega \setminus \{z_0\}$.

   c) Show that $F$ can be extended to $z_0$.

   d) Show that $F: \Omega \to \{z : |z| < 1\}$ is a bijection.

   e) Conclude that there exists an $\eta \in \mathbb{C}$, $|\eta| = 1$, such that $\eta F$ is the uniformization map.

4. Let $\Omega \subset \mathbb{C}$ be a domain bounded by a closed curve. Let $z_0 \in \Omega$. Let $u : \Omega \cup \partial \Omega \to \mathbb{R}$ be a continuous function such that

   - $u$ is harmonic in $\Omega$;
   - $u(z) = \log|z - z_0|$ for all $z \in \partial \Omega$.

   Let $G(z, z_0) := -\log|z - z_0| + u(z)$. Check that $G(z, z_0)$ is the Green function.