Möbius transformations.

Mapping of the form \( \frac{az + b}{cz + d} \), where \( a, b, c, d \in \mathbb{C} \) and \( ad - bc \neq 0 \), is a fractional linear transformation. These transformations can be composed as a sequence of simple transformations:

- translation \( f(z) = z + q, \ q \in \mathbb{C} \).
- homothety and a rotation \( f(z) = qz, \ q \in \mathbb{C} \).
- inversion (and reflection with respect to the real axis) \( f(z) = 1/z \).

1. Show that \( \frac{1}{z} \) is an inversion with respect to the unit circle.

**Definition:** A *generalized circle* is a straight line (on the complex plane extended by one point at infinity) or a circle.

2. Show that the action of the group of linear fractional transformations on generalized circles is transitive, that is, any generalized circle can be mapped into any other generalized circle.

3. **Preservation of angles.** Show that fractional linear transformations preserve angles.

4. **Cross-ratio preservation.** Show that cross-ratios are invariant under fractional linear transformations. That is, if a fractional linear transformation \( f \) maps four distinct points \( z_1, z_2, z_3, z_4 \) to points \( w_1 = f(z_1), w_2 = f(z_2), w_3 = f(z_3), w_4 = f(z_4) \) then:

\[
\frac{w_4 - w_1}{w_4 - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z_4 - z_1}{z_4 - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.
\]

**Corollary:** for any two ordered triples of distinct points, there is a unique fractional linear transformation that takes one triple to the other.

5. Let \( C_1, C_2 \) be two non-intersecting circles. Show that there exists a fractional linear transformation \( f \) such that \( f(C_1) \) and \( f(C_2) \) have centres at the same point.