Lozenge tilings on a cylinder

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Based on joint work with A. Ahn and R. Van Peski.
Ordinary partition

\[ \lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq 0) \]

\[ \lambda_i = 0 \quad \text{for} \quad i \gg 0 \]

The partition \[ \lambda = (8, 5, 4, 2, 2, 1) \]; \[ \lambda \leftrightarrow \{\lambda_i - i + \frac{1}{2}\} \]
partition

lozenge tiling

empty room

Plane partition  Skew plane partition  Cylindric partition
Random tiling/partition

Uniform measure: uniformly random tilings.

\( q^{\text{vol}} \) measure: \( \mathbb{P}[\text{tiling}] \propto q^{\text{vol}(\text{tiling})}, 0 < q < 1 \)
Cylindric partition

Let $q^N = t \in (0, 1)$, $q^{\text{vol}}$ measure on cylindric partitions: $\mathbb{P}(\lambda) \propto q^{\text{vol}}(\lambda)$. 
Lozenge tilings on a cylinder

lozenge tilings of the cylinder \( = \) shifted cylindric partitions

shift-mixed \( q^{\text{vol}} \) measure: \( \mathbb{P}(\lambda, S) \propto (u^S q^{NS^2}) q^{\text{vol}(\lambda)}, \)

\( u > 0, \ S \text{ is a vertical shift of the wall-floor interface} \)
The height function $H(\tau, y)$ vanishes for all sufficiently negative $y$ and $H(\tau, y) = y - S$ for all sufficiently large positive $y$.

The key questions: the large-scale behavior of

(a) the limit shape of the height function,
(b) fluctuations of the height function.
Limit shape

Let $q^N = t \in (0, 1)$

**Theorem (Ahn, R., Van Peski '21)**

*The height function $h_N$ of a $q^\text{vol}$-distributed cylindric partition of width $2N$ converges in probability to the following limit shape uniformly:*

$$\frac{1}{N} h_N(N\tau, Ny) \rightarrow H(y) = \begin{cases} 0 & y \leq \frac{\log 2}{\log t}, \\ \int_{\log 2 \over \log t}^{y} \frac{2 \arctan(\sqrt{4t^2 - 2u - 1})}{\pi} \, du & y \geq \frac{\log 2}{\log t}. \end{cases}$$

- [Borodin '07] showed result on local statistics which also computes the limit shape; our only real input here is showing concentration.
- The shift-mixed $q^\text{vol}$ measure has the same limit shape above, as the distribution of the shift is independent of the tiling and is finite-order independent of $N$. 
Theorem (Ahn, R., Van Peski ’21)

The fluctuations of the height function of a $q^{\text{vol}}$-distributed cylindric partition converges on the liquid region to the Gaussian free field in the Kenyon-Okounkov complex structure.

Theorem (Ahn, R., Van Peski ’21)

The fluctuations of the height function of a shift-mixed $q^{\text{vol}}$-distributed cylindric tiling are given by the same Gaussian free field with an additional discrete Gaussian shift component.
Simple random walk

- Limit shape:
  As \( X, T \to \infty \), \( \frac{X}{T} = \text{const} \), \( \frac{Z_{sT}}{X} \to s \) uniformly over \( s \in [0, 1] \).

- Fluctuations:
  \( \frac{Z_{sT} - \mathbb{E}[Z_{sT}]}{C \sqrt{T}} \to B_s \), where \( B_s \) is a standard Brownian bridge.

  \[
  G(s, s') := \text{Cov}(B_s, B_{s'}) = \min(s, s')(1 - \max(s, s'))
  \]

  is the Green’s function for Laplacian \( \Delta = \frac{\partial^2}{\partial s^2} \) on \([0, 1]\) with zero Dirichlet boundary conditions.
Gaussian Free Field

The Gaussian free field $\phi$ on $D$ is the random distribution such that pairings with test functions $\int_D f \phi$ are jointly Gaussian with covariance

$$\text{Cov} \left( \int_D f_1 \phi, \int_D f_2 \phi \right) = \int_{D \times D} f_1(z) G(z, w) f_2(w).$$

where $\phi$ is a conformally invariant random generalized function:

$$\Phi(z) = \sum_k \xi_k \frac{\phi_k(z)}{\sqrt{\lambda_k}},$$

[1d analog: Brownian Bridge]

where $\phi_k$ are eigenfunctions of $-\Delta$ on $D$ with zero boundary conditions, $\lambda_k$ is the corresp. eigenvalue, and $\xi_k$ are i.i.d. standard Gaussians.

The GFF is not a random function, but a random distribution.

GFF is a Gaussian process on $D$ with Green’s function of the Laplacian as the covariance kernel.
Conjecture [Kenyon-Okounkov ’05]
For lozenge tilings of simply connected planar regions, there exists a map $\zeta$ on liquid region $L$ so that

$$\sqrt{\pi}(H(x^\delta, y^\delta) - \mathbb{E}[H(x^\delta, y^\delta)]) \to \Phi \circ \zeta(x, y)$$

where $\Phi$ is the GFF and $\zeta$ is a local diffeomorphism onto its image.

Theorem  As mesh goes to zero, fluctuations of height $\Rightarrow$ Gaussian Free Field on $\mathbb{D}$ with zero boundary conditions.
Theorem (Kenyon-Okounkov '05)

In the liquid region (i.e. where $p_\triangle, p_\square, p_\Diamond > 0$), there exists a function $z(x, y)$ taking values in the upper half plane such that

$$\nabla \mathcal{H} = \frac{1}{\pi} (\arg z, - \arg(1 - z)) \quad \text{and} \quad \frac{-z_x}{1 - z} + \frac{z_y}{z} = 0.$$ 

Uniform measure: $\zeta = z$.

$q^\text{vol}$ measure (volume-constrain): let $q = e^{-c\delta}$, then $\zeta = e^{cx} z$. 
## Known results

<table>
<thead>
<tr>
<th>Limit shape</th>
<th>Fluctuations</th>
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<tbody>
<tr>
<td>[Cohn-Kenyon-Propp ’00] proved a.s. convergence to certain entropy-maximizers for uniformly random domino tilings of simply connected domains in $\mathbb{R}^2$.</td>
<td>Certain domains with no frozen regions (e.g. [Kenyon ’01], [R. ’18], [R. ’19]; [Kenyon ’08], [Berestycki-Laslier-Ray ’20]).</td>
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<tr>
<td>[Kenyon-Okounkov-Sheffield ’03] showed more generally (weighted doubly periodic bipartite dimer models on simply connected planar regions).</td>
<td>[Ahn ’20] $q^\text{vol}$ plane partitions.</td>
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<td>[Okounkov-Reshetikhin ’01] computed limit shape for $q^\text{vol}$ ordinary plane partitions.</td>
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<td>[Cerf-Kenyon ’01] Same limit shape for uniform measure on plane partitions of given volume.</td>
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Certain polygonal domains (e.g. [Borodin-Ferrari ’08], [Petrov ’12], [Bufetov-Knizel ’18]).

[Bufetov-Gorin ’17] Hexagon with a hole of fixed height (not simply connected).


Today: $q^\text{vol}$-distributed cylindric partitions and shift-mixed $q^\text{vol}$-distributed cylindric partitions.
### Model

<table>
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<tr>
<th>$q^{\text{vol}}$</th>
<th>shift-mixed $q^{\text{vol}}$</th>
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<tbody>
<tr>
<td><strong>measure supported on:</strong> cylindric partitions</td>
<td>shifted cylindric partitions $=$ lozenge tilings of the cylinder</td>
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<tr>
<td><img src="image1.png" alt="Diagram" /></td>
<td><img src="image2.png" alt="Diagram" /></td>
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<tr>
<td>$\mathbb{P}(\lambda) \propto q^{\text{vol}}(\lambda)$</td>
<td>$\mathbb{P}(\lambda, S) \propto u^S q^{\text{vol}}(\lambda, S) = (u^S q^{\text{vol}})^2 q^{\text{vol}}(\lambda)$, $u &gt; 0$, $S$ is a vertical shift of the wall-floor interface</td>
</tr>
<tr>
<td>periodic Schur process</td>
<td>shift-mixed periodic Schur process</td>
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<tr>
<td>determinantal structure, comes from the dimer model</td>
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</table>

$h(\tau, y) := \sum_{x < y} \left[ \text{there is no lozenge of type } \heartsuit \text{ at } (\tau, x) \right]$

$h(\tau, y)$ vanishes for all sufficiently negative $y$ and $h(\tau, y) = y - S$ for all sufficiently large positive $y$
Define a function $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ by
$$\mathcal{H}'(y) = \frac{2 \arctan \left( \sqrt{\frac{4}{t^2 y - 1}} \right)}{\pi} \mathbf{1}(0 < t^y < 2)$$
and $\lim_{y \to -\infty} \mathcal{H}(y) = 0$.

**Theorem (Ahn, R., Van Peski '21)**

The height function $\frac{1}{N} h_N$ of a $q^\text{vol}$ / shift-mixed $q^\text{vol}$-distributed cylindric partition of width $2N$ converges in probability to the limit shape $\mathcal{H}$ uniformly.

$p_\square = p_\Diamond$ (symmetry)

$$\mathcal{H}'(y) = 1 - p_\Diamond$$

$$\mathcal{L} = \{ (\tau, y) \in (0, 1] \times \mathbb{R} : 0 < t^{2y} < 4 \} = \{ (\tau, y) \in (0, 1] \times \mathbb{R} : y > \frac{\log 2}{\log t} \}.$$

$p_\square = p_\Diamond$ (symmetry)
Theorem (Ahn, R., Van Peski '21)

Fix $t \in (0, 1)$. Then the height function fluctuations of the unshifted $q^{\text{vol}}$ measure converges as $N \to \infty$ to the $\eta$-pullback of the Gaussian free field on the cylinder $C = (0, \frac{1}{2}) \times \mathbb{R}/\frac{\log t}{2\pi}$ with 0-Dirichlet boundary conditions, where $\eta : \mathcal{L} \to C$ is given by

$$\eta(\tau, y) = \frac{1}{2\pi i} \log \left( t^{\tau} \frac{2 - t^{2y} + i\sqrt{4t^{2y} - t^{4y}}}{2} \right).$$

Remark: $\eta$ defines the same conformal structure as the one conjectured by Kenyon-Okounkov.
A discrete Gaussian $S \sim \mathcal{N}_{\text{discrete}}(C, m)$ is the $\mathbb{Z}$-valued random variable defined by

$$\Pr(S = x) \propto e^{-C(x-m)^2}.$$ 

Theorem (Ahn, R., Van Peski '21)

Fix $u \in \mathbb{R}_{>0}$ and $t \in (0, 1)$, set $q := q(N) := t^{1/N}$. Then the height function fluctuations of the shift-mixed $q^\text{vol}$ measure converges to the $\eta$-pullback of the Gaussian free field with a discrete Gaussian shift $S \sim \mathcal{N}_{\text{discrete}}(\frac{|\log t|}{2}, \frac{\log u}{\log t}),$

$$h(2N\tau, 2Ny) - \mathbb{E}[h(2N\tau, 2Ny)] \xrightarrow{N \to \infty} \Phi(\eta(\tau, y)) - S\mathcal{H}'(y).$$
Methods

- $q^{\text{vol}}$ plane partitions are distributed as a certain Schur process [Okounkov-Reshetikhin '01]

- (shift-mixed) $q^{\text{vol}}$ cylindric partitions are certain (shift-mixed) periodic Schur process [Borodin '07]
Methods

- new formulas for joint exponential moments of the height function of periodic Schur processes
- similar formulas for the joint moments, which obtained formulas for observables for periodic Macdonald processes [Koshida '20]
- similar methods for GFF convergence for random matrices and random tilings used in e.g. [Borodin-Gorin '15], [Ahn '20]

**shift-mixed** $q^{\text{vol}}$: determinantal structure, Gaussian free field WITH an additional discrete Gaussian shift component

**unshifted** $q^{\text{vol}}$: NO determinantal structure, Gaussian free field
Holey hexagon

A domain topologically equivalent to the cylinder:

Height of hole depends on tiling. To choose random tiling either
★ allow hole height to vary
★ condition random tiling on fixed hole height

Analogy:
unrestricted tilings of cylinder ⇔ tilings of holey hexagon
unshifted cylindric partitions ⇔ tilings w/ fixed hole height.
Unshifted cylindric partitions $\leftrightarrow$ tilings w/ fixed hole height

Theorem (Bufetov-Gorin '17)

The uniform measure on tilings of the holey hexagon conditioned on fixed hole height has Gaussian free field fluctuations in Kenyon-Okounkov complex structure.

Theorem (Ahn, R., Van Peski '21)

The fluctuations of the height function of a $q^{\text{vol}}$-distributed cylindric partition of width $2N$ converges on the liquid region to the Gaussian free field in the Kenyon-Okounkov complex structure.
Dirichlet energy

Conjecture

For a general planar domain with a hole, the limiting fluctuations of the hole height are discrete Gaussian $N_{\text{discrete}}(C, m)$. Furthermore

$$C = \frac{\pi}{2} \int_{\zeta(L)} \|
abla g\|^2 \, dx \, dy \quad \text{(Dirichlet energy)}$$

of unique harmonic function $g$ which is 0 on outer boundary, 1 on inner boundary.

Rmk: To be proven for some domains in [Borot-Gorin-Guionnet, in prep.].
Unrestricted tilings of cylinder $\leftrightarrow$ tilings of holey hexagon

For shift-mixed $q^\text{vol}$ recall independent shift $S$ has

$$\Pr(S = x) \propto u^x q^{Nx^2}.$$ 

Equivalently (recall $t = q^N$)

$$S \sim \mathcal{N}_{\text{discrete}} \left( \frac{|\log t|}{2}, \frac{\log u}{\log t} \right)$$

and

$$C = \frac{|\log t|}{2}$$

is exactly the Dirichlet energy in previous conjecture for our case!
THANK YOU