EXIT DISTRIBUTIONS ASSOCIATED WITH LOOP-ERASED RANDOM WALKS AND RANDOM MATRICES

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22/04/2021

BACKGROUND ON THE 1-D CASE

u-dimensional Brownian motion started at $x_1 > x_2 > \ldots > x_n$ and conditioned to stay in the chamber $C = \{x \in \mathbb{R}^n : x_1 > x_2 > \ldots > x_n \}$ up to time $t > 0$.

The (unnormalized) transition density of this process is given by:

$$q_t(x, y) = \det_{n \times n} (p_t(x_i, y_j)) \quad , \quad x, y \in C$$

transition density of one-dimensional B.M

QUESTION: What is the distribution of the positions at time $t > 0$?
if the processes start at the origin?

\[
\lim_{x_i \to 0} \frac{1}{\text{Mx n}} \det (p_t(x_i, y_j)) = \frac{1}{\text{M}} e^{-\frac{1}{2} n \sum_{i=1}^{n} y_i^2} \prod_{1 \leq i \neq j \leq n} (y_i - y_j)
\]

\[\text{RHS} = \text{joint density of the eigenvalues of the Gaussian Orthogonal Ensemble (or GOE)}.
\]

= eigenvalues of a random (real) symmetric matrix:

\[
\begin{pmatrix}
N(0, t) & N(0, \frac{t}{2}) \\
N(0, \frac{t}{2}) & N(0, t)
\end{pmatrix}
\]

\[
\frac{n(n-1)}{2} \text{ entries}
\]

The main tool in the above treatment is the identity:

Korin-McGregor (also Gessel-Viennot), SO's

\[q_t(x, y) = \det (p_t(x_i, y_j)), \quad x, y \in \mathbb{C}.
\]

Proof: 1) Feynman-Kac (RHS solves the heat equation with b.c.)

2) Application of the reflection principle.

\[n=2
\]

\[
\det \begin{pmatrix}
p_t(x_1, y_1) & p_t(x_2, y_1) \\
p_t(x_2, y_1) & p_t(x_2, y_1)
\end{pmatrix} = p_t(x_1, y_1) p_t(x_2, y_1) - p_t(x_1, y_2) p_t(x_2, y_1)
\]

\[= \mathcal{P}(x_1 \rightarrow y_1) - \mathcal{P}(x_1 \rightarrow y_2) = (*).
\]
\[ (*) = \mathbb{P}(x_1 \rightarrow y_1, \text{ no intersection}) - \mathbb{P}(x_2 \rightarrow y_2, \text{ no intersection} ). \]

**Problem:** Two-dimensional (planar) processes

**Question:** What can we say if the processes under consideration are two-dimensional?

- Paths are allowed to have loops.
- Intersection in space, not necessarily in time.
- Analogue of the one-dimensional setting.

- Koralov-McGregor (Gessel-Viennot) is not applicable anymore.
- However, there is a generalisation for (discrete) planar processes by S. Fomin.


In a discretisation \( \Omega^\# \) of a planar domain \( \Omega \subset \mathbb{C} \), consider \( \xi_1, \xi_2, \ldots, \xi_n = n \) independent SRF on \( \Omega^\# \), then

\[ \mathbb{P}(\xi_i : x_i \rightarrow y_i, \xi_j \cap \xi_i = \emptyset, j \neq i) = \prod_{i=1}^{n} (\mathbb{P}(h(x_i, y_j))). \]
where
\[ h(x, y) = \mathbb{P}_x(\mathcal{E}_{12} = y) \] is the (discrete) Poisson kernel.

\[ n = 2 \]
\[ \mathcal{E}_{12} \cap \text{LE}(\mathcal{E}_1) = \emptyset \],
\[ \text{LE}(\mathcal{E}_1) = \text{Loop-erased random walk} \]

Fomin's reflection:

The symmetry in Fomin's non-intersecting condition is better explained in the context of branches of the (wired) UST = Uniform Spanning Tree.

That is (Dubédat, 2006):
\[ \mathbb{P}(\mathcal{E}_i : x_i \to y_j, \mathcal{E}_j \cap \text{LE}(\mathcal{E}_i) = \emptyset, j > i) = \mathbb{P}(n \text{ branches of the (wired) UST } x \to y) \]
**Connections with Random Matrices**

For $\Omega \subset \mathbb{C}$ simply connected and $x_1, \ldots, x_n, y_1, \ldots, y_1 \in \partial \Omega$, in a discretisation $\Omega \cap \mathbb{Z}^2$, $\delta > 0$ we have

\[
\det (h^\delta(x_i^\delta, y_j^\delta)) = \prod (x_i^\delta \rightarrow y_j^\delta, \xi_i^\delta \cap \mathcal{L}(\xi_j^\delta) = \emptyset, j > i)
\]

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\text{scaling limit } \delta \downarrow 0
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\[
\det (h(x_i, y_j)) \uparrow \text{Fomin's determinants}
\]

\text{(excursion) Poisson kernels of B.M.}

\[
\text{Distribution of the exit points:}
\]

\[
\leftrightarrow \text{random matrix ensembles.}
\]

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**Connections with Random Matrices**

1) (Planar) B.M. in the positive quadrant:

\[
h(x, y) = \frac{4xy}{\pi (x^2 + y^2)^2}, \quad y > 0, x > 0.
\]

If $0 < x_1 < \ldots < x_k$ and $0 < y_1 < \ldots < y_1$
\[
\lim_{x_i \to 1} \frac{1}{M_x} \det(h(x_i, y_j)) = \frac{1}{M} \prod_{j \leq \sigma} \prod_{i \leq \sigma \leq j} (y_i^2 - y_j^2)
\]

2) Half-unit disk:

\[
h(x_i, \theta) = \frac{1}{\pi} \frac{(1 - x_i^2) \sin \theta}{(1 - 2x_i \cos \theta + x_i^2)^2}, \quad |x_1| < 1, \quad 0 < \theta < \pi
\]

If \(-1 < x_1 < \ldots < x_n < 1\) and \(0 < \theta_1 = \ldots = \theta_n < \pi\)

\[
\lim_{x_i \to 0} \frac{1}{M_x} \det(h(x_i, \theta_i)) = \frac{1}{M} \prod_{i \leq j \leq \lambda} \sin \theta_j \prod_{i \leq j \leq \lambda} (\cos \theta_i - \cos \theta_j)
\]

\[
= \frac{1}{M} \prod_{i \leq j \leq \lambda} \sin \theta_j \prod_{i \leq j \leq \lambda} |e^{i\theta_i} - e^{i\theta_j}| |e^{i\theta_i} - e^{i\theta_j}|
\]

**Affine (Circular) Versions**

**Question:** What can we say if the domains under consideration are not simply connected? E.g., annulus.

Same problem: distribution of \(\theta_1, \theta_2, \ldots, \theta_n\) when the inner radius goes to zero.

- Discretise the model
- Add a zipper

We obtain an "affine" version of Fomin's formula for certain type of periodic (discrete) steps:

- J.A. O'Connell (2019)

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Proof. Reflection principle of Fomin on a careful selection of paths.

If we consider \( 0 < r < 1 \) in the RHS above, for a strip of width (radius) \( 0 < r < 1 \):

\[
h(\psi, 0) = \frac{\pi}{4\log r^2} \sech^2\left( \frac{\pi}{2\log r^1} (0 - \psi) \right), \quad -\pi \leq \psi, \psi < \pi.
\]
and we obtain, for appropriate initial and final points:

- J. A., O’Connell (2019)

For any $\bar{\nu}, \bar{\omega} \in \mathbb{C}$

$$\lim_{r \to 0} \frac{1}{M^m} \det \left( \sum_{k \in \mathbb{Z}} e^{2\pi i k} h(\nu_k, \omega_j + 2\pi k) \right) = \frac{1}{M} \prod_{1 \leq i < j \leq n} |e^{\Theta_i} - e^{\Theta_j}|.$$ 

This agrees with the eigenvalue distribution of the COE ensemble.

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**REFERENCES**


